

同步练习 P250 理学院 廖国勇 2016. 4. 13

一. 1. 以 2π 为周期的函数 $f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1+x^2 & 0 < x \leq \pi \end{cases}$ 的傅里叶级数在 $x=\pi$

处收敛于 _____

$$\begin{aligned} \text{解: } \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right) \Big|_{x=\pi} &= \left[\lim_{x \rightarrow -\pi^+} f(x) + \lim_{x \rightarrow \pi^-} f(x) \right] \cdot \frac{1}{2} \\ &= [-1 + 1 + \pi^2] \cdot \frac{1}{2} = \frac{\pi^2}{2} \quad (\text{狄利克雷定理 Dirichlet theorem}) \end{aligned}$$

2 设函数 $f(x) = \pi x + x^2$ ($-\pi < x < \pi$) 的傅里叶级数展开式为 $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, 则 $b_3 =$ _____

$$\begin{aligned} \text{解: } b_3 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin 3x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi x + x^2) \sin 3x dx \\ &= \pi \frac{2}{\pi} \int_0^{\pi} x \sin 3x dx \quad (\pi \sin 3x \text{ 为偶函数, } x^2 \sin 3x \text{ 为奇函数}) \\ &= 2 \cdot \frac{\pi}{2} \int_0^{\pi} \sin 3x dx \\ &= -\frac{\pi}{3} \cos 3x \Big|_0^{\pi} \\ &= \frac{2}{3} \pi \end{aligned}$$

注, 翻转公式 $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$, 于是

$$\begin{aligned} I &= \int_0^{\pi} x \sin 3x dx = \int_0^{\pi} (0+\pi-x) \sin 3(0+\pi-x) dx \\ &= \int_0^{\pi} (\pi-x) \sin 3x dx = \pi \int_0^{\pi} \sin 3x dx - \int_0^{\pi} x \sin 3x dx \\ &= \pi \int_0^{\pi} \sin 3x dx - I \\ \Rightarrow I &= \frac{\pi}{2} \int_0^{\pi} \sin 3x dx \end{aligned}$$

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二. 2 将函数 $f(x) = \cos \frac{x}{2}$ 在 $[0, \pi]$ 上展开成余弦级数.

解: $a_0 = \frac{2}{\pi} \int_0^{\pi} \cos \frac{x}{2} dx = \frac{4}{\pi} \sin \frac{x}{2} \Big|_0^{\pi} = \frac{4}{\pi}$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos \frac{x}{2} \cdot \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} (\cos \frac{1+2n}{2} x + \cos \frac{1-2n}{2} x) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\cos \frac{1+2n}{2} x + \cos \frac{1-2n}{2} x) dx$$

$$= \frac{1}{\pi} \cdot \left(\frac{2}{1+2n} \sin \frac{1+2n}{2} x + \frac{2}{1-2n} \sin \frac{1-2n}{2} x \right) \Big|_0^{\pi}$$

$$= \frac{1}{\pi} \left(\frac{2}{1+2n} (-1)^n + \frac{2}{1-2n} (-1)^n \right)$$

$$= \frac{4}{\pi} \frac{(-1)^{n+1}}{4n^2-1} \quad (n=1, 2, \dots, \infty)$$

于是, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^{n+1}}{\pi (4n^2-1)} \cos nx \quad (0 \leq x \leq \pi)$$

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3. 将函数 $f(x) = \begin{cases} 2x+1, & -3 \leq x < 0 \\ 1, & 0 \leq x < 3 \end{cases}$ 在一个周期内展开成傅里叶级数

$$\text{解: } a_0 = \frac{1}{3} \int_{-3}^3 f(x) dx = \frac{1}{3} \int_{-3}^0 (2x+1) dx + \frac{1}{3} \int_0^3 1 dx$$
$$= \frac{1}{3} (x^2+x) \Big|_{-3}^0 + \frac{1}{3} x \Big|_0^3 = \frac{1}{3} (-1)(9-3) + 1 = -1$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx = \frac{1}{3} \int_{-3}^3 f(x) \cos \frac{n\pi x}{3} dx \quad (n=1, 2, \dots)$$

$$= \frac{1}{3} \int_{-3}^0 (2x+1) \cos \frac{n\pi x}{3} dx + \frac{1}{3} \int_0^3 \cos \frac{n\pi x}{3} dx$$

$$= \frac{2}{3} \int_{-3}^0 x \cos \frac{n\pi x}{3} dx = \frac{2}{3} \cdot \frac{3}{n\pi} \int_{-3}^0 x d \sin \frac{n\pi x}{3}$$

$$= -\frac{2}{n\pi} \int_{-3}^0 \sin \frac{n\pi x}{3} dx = \frac{6}{\pi^2 n^2} \cos \frac{n\pi x}{3} \Big|_{-3}^0 = \frac{6}{\pi^2 n^2} (1+(-1)^n)$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{1}{3} \int_{-3}^0 (2x+1) \sin \frac{n\pi x}{3} dx + \frac{1}{3} \int_0^3 \sin \frac{n\pi x}{3} dx$$

$$= \frac{1}{3} \int_{-3}^0 2x \sin \frac{n\pi x}{3} dx = \frac{2}{3} \cdot \frac{3}{n\pi} \int_{-3}^0 x d(-\cos \frac{n\pi x}{3})$$

$$= \frac{2}{n\pi} \left[(-x \cos \frac{n\pi x}{3}) \Big|_{-3}^0 + \int_{-3}^0 \cos \frac{n\pi x}{3} dx \right]$$

$$= \frac{2}{n\pi} \cdot 3 \cdot (-1)^{n+1} = \frac{6}{n\pi} (-1)^n \quad n=1, 2, \dots$$

$$\text{于是, } f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \left[\frac{6 [1+(-1)^n]}{n^2 \pi^2} \cos \frac{n\pi x}{3} + \frac{6}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{3} \right]$$

$$(-3 \leq x < 3)$$

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一. 3. 以2为周期的函数 $f(x) = \begin{cases} 2, & -1 < x \leq 0 \\ x^2, & 0 < x \leq 1 \end{cases}$ 的傅里叶级数在 $x=1$

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解: $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \Big|_{x=1} = \lim_{x \rightarrow 1^+} f(x) + \lim_{x \rightarrow 1^-} f(x) = \frac{2+1^2}{2} = \frac{3}{2}$

4. 函数 $f(x) = x^2$, $0 \leq x \leq 1$, 设 $S(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$, $-\infty < x < +\infty$

其中 $b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx$ ($n=1, 2, \dots$), 则 $S(-\frac{1}{2})$

解: $S(x)$ 中不含余弦函数项, 故 $f(x) = x^2$ 做奇延拓后得到的函数

的傅里叶级数为 $S(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$.

$$f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ -x^2 & -1 < x < 0 \end{cases}$$

$$\text{故 } S(-\frac{1}{2}) = f(-\frac{1}{2}) = -(-\frac{1}{2})^2 = -\frac{1}{4}.$$

二. 1. 将以 2π 为周期的函数 $f(x) = x$ ($-\pi \leq x < \pi$) 展开成傅里叶级数

解: $f(x)$ 为奇函数, 故 $a_n = 0$ ($n=0, 1, 2, \dots$)

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x d(-\frac{1}{n} \cos nx) dx = \left[\frac{2}{\pi} x \cdot (-\frac{1}{n} \cos nx) \right]_0^{\pi} + \frac{2}{n\pi} \int_0^{\pi} \cos nx dx$$

$$= \frac{2}{\pi} \left(-\frac{\pi}{n} \right) + \frac{2}{n\pi} \sin nx \Big|_0^{\pi} = \frac{2}{n} (-1)^{n+1} \quad (n \neq 0, \text{ 即 } n=1, 2, 3, \dots)$$

$$\text{于是, } f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (\sin nx) (-1)^{n+1} \quad (x \in \mathbb{R})$$

注, 该题参考答案 $2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi} \sin nx$ 是错的。

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5. 将函数 $f(x) = 2 + |x|$ ($-1 \leq x \leq 1$) 展开成以 2 为周期的傅里叶级数, 并求级数 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 的和.

解: $f(x)$ 是偶函数, 于是 $b_n = 0$, $n = 1, 2, 3, \dots$

$$a_0 = 2 \int_0^1 f(x) dx = 2 \int_0^1 (2+x) dx = 5$$

$$a_n = 2 \int_0^1 (2+x) \cdot \cos n\pi x dx = 2 \int_0^1 2 \cos n\pi x dx + 2 \int_0^1 x \cos n\pi x dx$$

$$2 \int_0^1 2 \cos n\pi x dx = \frac{4}{n\pi} \sin n\pi x \Big|_0^1 = 0$$

$$2 \int_0^1 x \cos n\pi x dx = \frac{2}{n\pi} \int_0^1 x d \sin n\pi x = \frac{2}{n\pi} x \sin n\pi x \Big|_0^1 - \frac{2}{n\pi} \int_0^1 \sin n\pi x dx$$

$$= -\frac{2}{n\pi} \int_0^1 \sin n\pi x dx = \frac{2}{n\pi} \cdot \frac{1}{n\pi} \cos n\pi x \Big|_0^1 = \frac{2}{n^2\pi^2} [(-1)^n - 1]$$

$$\text{所以 } a_n = \frac{2}{n^2\pi^2} [(-1)^n - 1] \quad n = 1, 2, 3, \dots$$

$$f(x) = 2 + |x| = \frac{5}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} [(-1)^n - 1] \cos n\pi x \quad (x \in \mathbb{R})$$

$$\text{特别地, 令 } x=0, \text{ 有 } f(0) = 2 + |0| = \frac{5}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} [(-1)^n - 1] \cos n \cdot 0$$

$$\text{即 } \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right) = \frac{\pi^2}{16} \cdot \left(\frac{1}{2} \right)^{-1}$$

$$\text{设 } A = \sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ 则 } A = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n-1)^2} + \dots \right) +$$

$$\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots + \frac{1}{(2n)^2} + \dots \right)$$

$$= \frac{\pi^2}{16} \cdot \frac{1}{4} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right)$$

$$= \frac{\pi^2}{16} \cdot \frac{1}{4} \cdot A$$

$$\Rightarrow A = \frac{\pi^2}{12} \cdot \left(\frac{1}{2} \right)^{-1} = \frac{\pi^2}{6}$$

$$\text{注, } \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} A$$